Survey of the known algebraic solutions of Painlevé VI

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Classical example

Icosahedral rotation group A_5 of order 60

$$A_5 = \Delta_{235} = \langle a, b, c \mid a^2 = b^3 = c^5 = abc = 1 \rangle$$

- natural to look for ODEs on $\mathbb{P}^1 \setminus 3$ points with monodromy A_5

$$A_5 \subset SO_3(\mathbb{R}) \subset SO_3(\mathbb{C}) \cong PSL_2(\mathbb{C})$$

- look for connections on rank 2 vector bundles with projective monodromy A_5

Schwarz's List (1873)

Gauss hypergeometric equation

$\Longrightarrow$ logarithmic connection $\left(\frac{A_1}{z} + \frac{A_2}{z-1}\right) dz$

- A_1, A_2 2×2 rank 1 matrices
  { (twist by log connection on line bundle)
- $A_1, A_2 \in sl_2(\mathbb{C})$ (2×2 trace free)

[connection on trivial principal $SL_2(\mathbb{C})$ bundle over $\mathbb{CP}^1$]

Algebraic horizontal sections classified by Schwarz

$\rightsquigarrow$ List with 15 entries:

1 dihedral family
2 Tetrahedral solutions
2 Octahedral solutions
10 Icosahedral solutions

[Rigid]

régulier. D'ailleurs, si l'intégrale générale de l'une des équations $\mathcal{E}(\lambda, \mu, \nu)$, $\mathcal{E}(1-\lambda, 1-\mu, \nu)$, $\mathcal{E}(\lambda, 1-\mu, 1-\nu)$, $\mathcal{E}(1-\lambda, \mu, 1-\nu)$ est une fonction algébrique de x, il en est évidemment de même des trois autres (n° 35).

Soient $\lambda'\pi$, $\mu'\pi$, $\nu'\pi$ les angles de celui des quatre triangles PQR, PQ'R, QP'R, P'Q'R pour lequel la somme des angles est la plus petite, les nombres λ', μ', ν' étant rangés par ordre de grandeur décroissante. Pour que l'intégrale générale de $E(\alpha, \beta, \gamma)$ soit une fonction algébrique, il faut et il suffit que les nombres λ', μ', ν', qui se déduisent de α, β, γ comme il a été expliqué, figurent dans le tableau ci-dessous de Schwarz :

	λ'	μ'	ν'		
(I)	$\frac{1}{2}$	$\frac{1}{2}$	»		double pyramide
(II)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$		Tétraèdre
(III)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		Tétraèdre
(IV)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$		Cube et octaèdre
(V)	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$		Cube et octaèdre
(VI)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	a b c	Icosaèdre et dodécaèdre
(VII)	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	b b d	Icosaèdre et dodécaèdre
(VIII)	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	b c c	Icosaèdre et dodécaèdre
(IX)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	a c d	Icosaèdre et dodécaèdre
(X)	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	b c d	Icosaèdre et dodécaèdre
(XI)	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	d d d	Icosaèdre et dodécaèdre
(XII)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	b b c	Icosaèdre et dodécaèdre
(XIII)	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	c c c	Icosaèdre et dodécaèdre
(XIV)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	a b d	Icosaèdre et dodécaèdre
(XV)	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	b c d	Icosaèdre et dodécaèdre

A_5 conjugacy classes

a $\frac{1}{2}$-turn
b $\frac{1}{3}$-
c $\frac{1}{5}$-
$d=c^2$ $\frac{2}{5}$-

Naive generalisations

One more pole :- $\sum_1^3 \frac{A_i}{z-a_i}\,dz$ WLOG $a_1, a_2, a_3 = 0, t, 1$

(A) $A_i \in sl_2(\mathbb{C})$

(B) A_i 3×3 rank 1

[Both minimally non-rigid – 2d moduli spaces]

Qn Analogue of Schwarz's list for these?

- can now answer this "nonrigid Schwarz list"

- still linear

Example of problem B:

Full symmetry group - icosahedral reflection group (order 120)

$$H = \left\langle r_1, r_2, r_3 \;\middle|\; \begin{matrix} r_1^2 = r_2^2 = r_3^2 = 1 \\ (r_1 r_2)^2 = (r_2 r_3)^3 = (r_3 r_1)^5 = 1 \end{matrix} \right\rangle$$

$$\subset O_3(\mathbb{R}) \subset GL_3(\mathbb{C})$$

- look for connections on rank 3 bundles / $\mathbb{P}^1 \setminus 4$ points with monodromy H (generated by 3 reflections)

- (essentially) solved around 1997 by Dubrovin - Mazzocco

  3 inequivalent triples of generating reflections

  1 ~ K. Saito's icosahedral Frobenius manifold

  1 involves 10 pages of 40 digit integers

$$-1226684412907984419281022032089194096771900\,t^{8}$$
$$-1114701349894370233505605371103641055314707\,t^{9}$$
$$+706698148832598485833137372995728746006888\,t^{10}$$
$$-230885597278675059768074093486733449982986\,t^{11}$$
$$+40110760213781966306595755424591426952408\,t^{12}$$
$$-2944406938738808019234484282441173992613\,t^{13}$$
$$+29909989810256194655311832623132956\,t^{14})\; x\,(1+x)\,y^{15}$$
$$+3\;(-1934531152410368929980642986659558434434933$$
$$+1658808400180625178945241486611794108530725 46\,t$$
$$-4339753511866615278991905104198610316815772 23\,t^{2}$$
$$+5155163066743090517140960864920723318089180 60\,t^{3}$$
$$-2835628767616075959790243437839552709908522 89\,t^{4}$$
$$+3508971787065203716652886578207124222849187 34\,t^{5}$$
$$+3329792899012718704983130445738794368757890 9\,t^{6}$$
$$-1291776424485166487282762047255608280322685 6\,t^{7}$$
$$-2667136232453283569559792524882581432924 63\,t^{8}$$
$$+5559001988444403518149870305222631626523 34\,t^{9}$$
$$+3448091251995758234969231253855658313155 95\,t^{10}$$
$$-3256890724598070084571219080753719914837 16\,t^{11}$$
$$+1173884397830202068948971444600708463329 49\,t^{12}$$
$$-2112368807268636856817019649675393743718 2\,t^{13}$$
$$+15691615887424347602822354800901000822 55\,t^{14})\; x\,y^{16}$$
$$+3\;(978329976048894803021943300608357029668935 7$$
$$-5932111934791854365993067652198438404216943 0\,t$$
$$+1414164778375296517266862645727728221934300 55\,t^{2}$$
$$-1770968098782894567939037963774764552576735 00\,t^{3}$$
$$+1279075864796514223185644108359081927867633 65\,t^{4}$$
$$-5437265830913964073343929602104804972674669 8\,t^{5}$$
$$+1348839437598325917838626903107782654132367 9\,t^{6}$$

Dubrovin – Mazzocco
Preprint ~ 1997

Nonlinear analogue — Painlevé VI equation

Explicit form of simplest (abelian) Gauss-Manin systems are Gauss hypergeometric equations [periods of families of elliptic curves - Gauss]

Explicit form of simplest non-abelian Gauss-Manin connection is the Painlevé VI equation

- "nonlinear" analogue of hgeom. equation
- solutions branch at $0, 1, \infty \in \mathbb{P}^1$ (still)

Main question: Analogue of Schwarz's list for P_{VI}?

C

- still open
- will describe what is known + methods used

Other motivations

- often geometrically significant
  e.g. link with: Frobenius manifolds
  Poncelet problem / modular curves
  Elliptic fibrations

- Method to construct non-rigid Fuchsian systems with known monodromy (Riemann-Hilbert problem)

- P_{VI} is reduction of 4d ASDYM equations
  - so any solution should be interesting

- Good 'testing ground' for general Painlevé VI machinery

What is Painlevé VI ?

- explicit form of simplest non-Abelian Gauss–Manin connection

- equation controlling isomonodromy deformations of certain log connections/Fuchsian systems on $\mathbb{P}^1$

- most general 2nd order ODE with Painlevé property

- certain dimensional reduction of ASDYM equations

- equation related to certain elliptic integrals with moving endpoints (R. Fuchs/Manin)

The Painlevé VI equation (P_{VI}):

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

where the constants $\alpha, \beta, \gamma, \delta$ are related to the parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$ by:

$$\alpha = (\theta_4 - 1)^2/2, \qquad \beta = -\theta_1^2/2, \qquad \gamma = \theta_3^2/2, \qquad \delta = (1 - \theta_2^2)/2.$$

Main properties (well-known)

- Painlevé property – critical singularities at $0, 1, \infty$:
  "Any local solution $y(t)$ near $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ extends to meromorphic function on universal cover"

- Trichotomy
  Any solution is either
  - a 'new' transcendental function
  - a solution of a 1st order Riccatti eqn
  - an algebraic function

- Governs isomonodromic deformations of rank 2 log. connections $\sum_1^3 \frac{A_i}{z - a_i}$ on $\mathbb{P}^1$ (type A)
  Eigenvalues $A_i = \pm\theta_i/2$ $\qquad (A_4 = -\sum_1^3 A_i)$

- $W_{aff}(F_4)$ symmetry group (standard action on $\mathbb{C}^4 \ni (\theta_1, \theta_2, \theta_3, \theta_4)$)

Definition

An algebraic solution to P_{VI} is an irreducible polynomial $F(y,t) \in \mathbb{C}[y,t]$ s.t the algebraic function $y(t)$ defined implicitly by

$$F(y(t),t) = 0$$

solves P_{VI} for some value of the parameters

Definition'

... is an (irreducible compact) algebraic curve Π and two rational functions $y, t : \Pi \to \mathbb{P}^1$ s.t

① t is a Belyi map (branch locus $\subset \{0,1,\infty\}$)

② $y(t)$ solves P_{VI} (for some parameters)

Def^n

Π is a minimal Painlevé curve (or an "efficient parameterisation") if

$$\Pi \underset{\text{birational}}{\cong} \{ F(y,t) = 0 \}$$

Principal Invariants

- degree = degree of Belyi map $t : \Pi \to \mathbb{P}^1$ (if Π minimal)
  = $\deg_y(F)$ = "no. of branches"
- genus = genus(Π) (if minimal)
  = genus of function field $\mathbb{C}(y, t)$

Basic examples of algebraic solutions to Painlevé VI (Hitchin, Dubrovin):

Three-branch tetrahedral solution:

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/3, 1/3, 1/3, 2/3)$$

Four-branch dihedral solution:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 1/2)$$

Four-branch octahedral solution:

$$y = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/4, 1/4, 1/4, 1/4)$$

Basic *families* of algebraic solutions to Painlevé VI:

Square root family:

$$y = \pm\sqrt{t}$$

$$\theta_2 = \theta_3 \quad \text{and} \quad \theta_1 + \theta_4 = 1$$

Three-branch tetrahedral family:

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}$$

$$\theta_1/2 = \theta_2 = \theta_3, \theta_4 = \frac{2}{3}$$

Four-branch dihedral family:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1}$$

$$\theta_1 = \theta_2 = \theta_3, \theta_4 = 1/2$$

Four-branch octahedral family:

$$y = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}$$

$$\theta_1 = \theta_2 = \theta_3, \theta_4 = 1 - 3\theta_1$$

Okamoto's affine Weyl group action

If $y(t)$ solves P_{VI} with parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$

then $y(t)$ ———————————— $(-\theta_1, \theta_2, \theta_3, \theta_4)$

& $y(t)$ ———————————— $(\theta_1, -\theta_2, \theta_3, \theta_4)$

& $y(t)$ ———————————— $(\theta_1, \theta_2, -\theta_3, \theta_4)$

& $y(t)$ ———————————— $(\theta_1, \theta_2, \theta_3, 2-\theta_4)$

- reflections in hyperplanes $\theta_i = 0$ $(i=1,2,3)$, $\theta_4 = 1$

Thm If defined, $y + \phi/x$ solves P_{VI} with params

$$(\theta_1 - \phi, \theta_2 - \phi, \theta_3 - \phi, \theta_4 - \phi) \quad \text{where } \phi = \textstyle\sum_1^4 \theta_i/2$$

$$\& \quad x = \frac{1}{2}\left(\frac{(t-1)y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y - t} - \frac{ty' + \theta_3}{y - 1}\right).$$

- reflection in hyperplane $\sum \theta_i = 0$
- these 5 transformations generate a group isom. to $W_{aff}(D_4)$

$\hat{D}_4 =$ 

- Can add in S_4 syms of $\hat{D}_4$ (R. Fuchs / Schlesinger) & get sym. group $\cong W_{aff}(F_4)$

$W_{aff}(F_4)$ is an infinite group

$\cong \quad W(F_4) \ltimes \Lambda(F_4)$

translation subgroup $\cong \mathbb{Z}^4$

finite reflection group (order 1152)

$\Lambda(F_4) = \langle \varepsilon_i \pm \varepsilon_j \rangle_{\mathbb{Z}}$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ standard ON basis of $\mathbb{C}^4$

(coords θ_i on $\mathbb{C}^4$ so $\sum \theta_i \varepsilon_i \in \mathbb{C}^4$)

$W_{aff}(F_4)$ generated by reflections in the five

hyperplanes: $\theta_2 = \theta_3$, $\theta_3 = \theta_4$, $\theta_4 = 0$, $\theta_1 = \theta_2 + \theta_3 + \theta_4$

$\theta_1 + \theta_2 = 1$

Shape of table so far

[Riccatti / Rational solutions Watanabe, Mazzocco, Yuan-Li]

4 continuous families $g=0$ $d=$ 2 $\sqrt{t}$ (Picard/Okamoto 3)
3
4 } Hitchin / Dubrovin
4

1 discrete family $\theta = (0001) \sim (\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ Dihedral ~ Poncelet problems
(Picard, R. Fuchs, Hitchin) d, g unbounded

45 exceptional solutions:

		d	g	
1	Tetrahedral	6	0	Andreev–Kitaev
7	Octahedral	6–16	0,1	(2 by Kitaev)
33	Icosahedral	5–72	0,1,2,3,7	1 by Dubrovin 2 by Dub.–Mazzocco 2 by Kitaev
1	'Klein'	7	0	
3	'237'	18	1	(1 by Kitaev)

Shape of table so far

[Riccatti / Rational solutions   Watanabe, Mazzocco, Yuan-Li]

4 continuous families $g=0$ $d=$ 2, 3, 4, 4

- 2: $\sqrt{t}$ (Picard/Okamoto 3)
- 3, 4, 4: Hitchin / Dubrovin

1 discrete family $\theta = (0001) \sim (\tfrac12 \tfrac12 \tfrac12 \tfrac12)$ Dihedral ~ Poncelet problems
(Picard, R. Fuchs, Hitchin) d, g unbounded

30 ~~45~~ exceptional solutions: modulo quadratic transformations

			d	g	
0	~~1~~	Tetrahedral	6	0	~~Andreev-Kitaev~~
2	~~3~~	Octahedral	6–16	0,1	1 (by Kitaev)
24	~~33~~	Icosahedral	5–72	0,1,2,3,7	1 by Dubrovin; 2 by Dub.–Mazzocco; 2 by Kitaev
1	~~2~~	'Klein'	7	0	
3	~~3~~	'237'	18	1	(1 by Kitaev)

Construction problem divides in two (roughly speaking):

(a) Finding solutions topologically

(b) Constructing topological solutions explicitly

Methods:

(a) 1. Finite monodromy groups $\begin{cases} SL_2 \\ GL_3 \end{cases}$

2. Pullbacks

(b) 1. Algebraic geometry [Twistors, Poncelet, Frobenius Mfds, Elliptic fibrations ...]

2. Pullbacks

3. Asymptotics (à la Jimbo)

Relations $P_{VI} \leftrightarrow$ Schlesinger's equations

[Two Fuchsian Lax pairs]

Suppose $A_1(t)$, $A_2(t)$, $A_3(t)$ solve Schlesingers eqns

$$\frac{dA_1}{dt} = \frac{[A_2, A_1]}{t}, \quad \frac{dA_3}{dt} = \frac{[A_2, A_3]}{t-1}, \quad \underset{\text{constant}}{\Sigma_1^3 A_i}$$

then ⓪ The linear connection

$$A = \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1} \right) dz$$

varies isomonodromically

and (if $\Sigma_1^3 A_i$ diagonal)

① If $A_i \in sl_2(\mathbb{C})$, the value $y(t)$ of z where

(Jimbo–Miwa) $\left(z(z-t)(z-1)A \right)_{12}$ is zero

solves P_{VI} with parameters $\underset{\sim}{\theta}$ s.t.

A_i has evals $\pm\theta_i/2$, $\Sigma_1^3 A_i = \begin{pmatrix} -\theta_4 & \\ & \theta_4 \end{pmatrix} / 2$

② If A_i 3×3 rank 1 then the value $y(t)$ of z where $\left(z(z-t)(z-1)A\right)_{23}$ is zero solves P_{VI} with parameters

$$\underset{\sim}{\theta} = (\lambda_1-\mu_1, \lambda_2-\mu_1, \lambda_3-\mu_1, \mu_3-\mu_2)$$

where $\lambda_i = \mathrm{Tr}(A_i)$ $i=1,2,3$

$$\Sigma_1^3 A_i = \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} \quad \text{so } \Sigma\lambda_i = \Sigma\mu_i$$

[Proc LMS (3) 90, 2005]

Note Positions of zeros of other 5 off diag. entries also solve P_{VI} — by ②, conjugate A by permutation matrix

- params change by corresponding permutation of μ's
- observe swapping μ_1 & μ_3 ⇒

$$\underset{\sim}{\theta} \mapsto (\theta_1-\phi, \theta_2-\phi, \theta_3-\phi, \theta_4-\phi)$$

$$\phi = \mu_3-\mu_1 = \Sigma_1^4\theta_i/2$$

- get direct geometrical interpretation of main Okamoto trfm.

If $y(t)$ solves $\mathrm{P_{VI}}$ with parameters

$$\theta_1 = \lambda_1 - \mu_1, \quad \theta_2 = \lambda_2 - \mu_1, \quad \theta_3 = \lambda_3 - \mu_1, \quad \theta_4 = \mu_3 - \mu_2$$

and we define $x(t)$ via

$$x = \frac{1}{2}\left(\frac{t(t-1)y'}{y(y-1)(y-t)} - \frac{\theta_1}{y} - \frac{\theta_3}{y-1} - \frac{\theta_2+1}{y-t}\right)$$

then the family of Fuchsian systems

$$\frac{d}{dz} - \left(\frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}\right)$$

will be isomonodromic as t varies, where

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

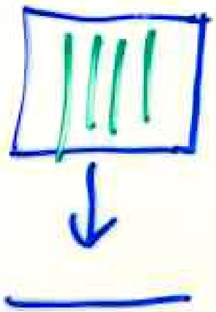
$$b_{12} = \lambda_1 - \mu_3 y + (\mu_1 - xy)(y-1), \qquad b_{32} = (\mu_2 - \lambda_2 - b_{12})/t,$$

$$b_{13} = \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), \qquad b_{23} = (\mu_2 - \lambda_3)t - b_{13},$$

$$b_{21} = \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, \qquad b_{31} = (\mu_2 - \lambda_1 - b_{21})/t.$$

[6 results on P_6, math.AG/0503043]

Aside on flat connections



$$\begin{array}{c} M \\ \pi \downarrow \\ B \end{array}$$

Fibre bundle, std fibre F

locally a product

$M_b = \pi^{-1}(b) \cong F \quad (\forall\, b \in B)$

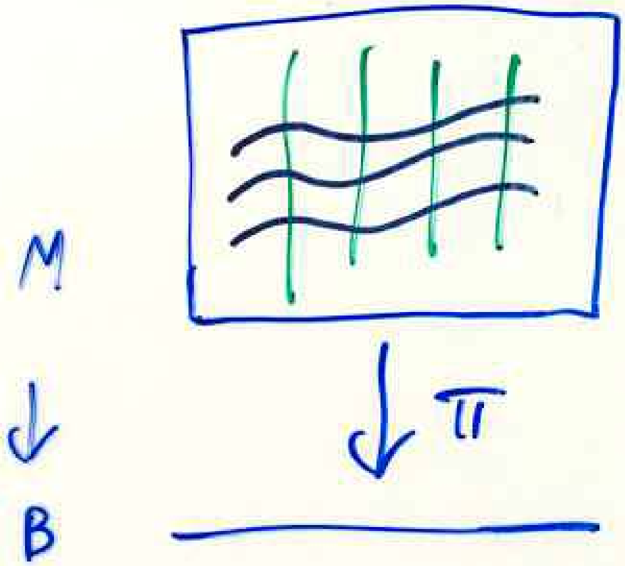
small $U \subset B \Rightarrow \pi^{-1}(U) \cong \underbrace{U \times F}_{\text{local trivialization}}$

- Complete flat connection on $M \to B$ is a way to identify 'nearby' fibres canonically:

  If $U \subset B$ open ball (i.e contractible) get isomorphisms $M_{b_1} \cong M_{b_2} \quad \forall\, b_1, b_2 \in U$

  – so get natural choice of local trivialization$/U$

- The connection is the infinitesimal object giving these isomorphisms



Horizontal lines
(sections of π)
~ identifications of fibres
– determined by their tangent spaces
$H_p \subset T_p M$ (pen

If $p \in M$ have vertical subspace

$$V_p = T_p M_{\pi(p)}$$

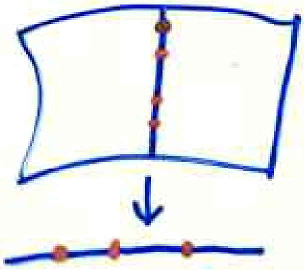
– connection is choice of field of horizontal subspaces H_p transverse to V_p

$$T_p = V_p \oplus H_p$$

– choose coords on fibres & base $\Rightarrow$
1st order system of coupled (nonlinear) differential equations

Conceptual approach

Consider universal family of $\mathbb{P}^1$'s with 4 punctures (ordered)



$$C \hookleftarrow F \cong \mathbb{P}^1 \setminus 4\text{ points}$$
$$\downarrow$$
$$B := \mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

- Replace each fibre F by $H^1(F, G)$, $G = SL_2(\mathbb{C})$

Two viewpoints here on H^1 :-

Betti Moduli of π_1 representations $\quad \mathrm{Hom}(\pi_1(F), G)/G$

$\wr\Uparrow$ Riemann-Hilbert

DeRham Moduli of connections on holomorphic vector bundles

- Get two (nonlinear) fibrations over $B = \mathcal{M}_{0,4}$
- As in abelian case get flat connection on bundle (now nonlinear connection)

$$
\begin{tikzcd}
\mathcal{M}_{DR} \arrow[r, "RH"] \arrow[d] & \mathcal{M}_{Betti} \arrow[d] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

Two descriptions of connection:

- Betti (periods $\rightsquigarrow$ monodromy)

  - keep monodromy representation constant

- DeRham $\begin{pmatrix} \text{one forms} \rightsquigarrow \text{connections on vector bundles} \\ \text{closed} \rightsquigarrow \text{flat} \end{pmatrix}$

  - extend flat connection on fibre $\mathcal{Y}$ to full flat connection on family of fibres & restrict to another fibre

Applications DR $\rightsquigarrow$ explicit nonlinear equations $\rightsquigarrow$ P_{VI}

Betti $\rightsquigarrow$ explicit description of monodromy of nonlinear connection

Explicit nonlinear equations

$\mathcal{M}_{DR}$ well approximated by moduli of log. connections on trivial bundles over $\mathbb{P}^1$:-

$$\mathcal{M}^* \cong \left\{ d - \sum_1^3 \frac{A_i}{z-a_i} dz \right\} / \text{isomorphism}$$

$$\cong \left\{ (A_1, \ldots, A_4) \,\middle|\, A_i \in \mathfrak{g}, \ \sum_1^4 A_i = 0 \right\} / G \ \times B$$

Nonlinear connection on $\begin{array}{c}\mathcal{M}^*\\ \downarrow \\ B\end{array}$ was computed by Schlesinger:

Horizontal sections satisfy $\quad \dfrac{\partial A_i}{\partial a_j} = \dfrac{[A_i, A_j]}{a_i - a_j} \quad i \neq j$

~ flatness of full connection $\quad d - \sum A_i \dfrac{d(z-a_i)}{z-a_i}$

Note
- fibres of $\mathcal{M}^*/B$ 6d Poisson manifolds
- Schlesinger's equations preserve the eigenvalues of each A_i $(i=1,2,3,4)$
- flows restrict to 2d symplectic leaves
- choose coords x, y on leaves $\Rightarrow$ coupled 1st order ODEs
- eliminate $x \Rightarrow$ 2nd order ode for $y(t)$ – Painlevé VI (t = coord on $B = \mathcal{M}_{0,4}$)

Monodromy of Painlevé VI

~ monodromy of connection on $\begin{array}{c} M_{Betti} \\ \downarrow \\ B \end{array}$

- connection is complete & flat so $\Leftrightarrow$

  action of $\pi_1(B) \cong F_2 \circlearrowleft$ fibre M_t

  $$M_t = \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus 4\text{points}), G)/G$$

Given choice of loops generating $\pi_1(\mathbb{P}^1 \setminus 4\text{points})$ get

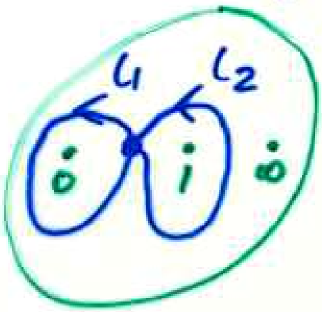
$$M_t \cong \{(M_1 M_2 M_3 M_4) \mid M_i \in G, M_4 M_3 M_2 M_1 = 1\}/G$$
$$\cong G^3/G$$

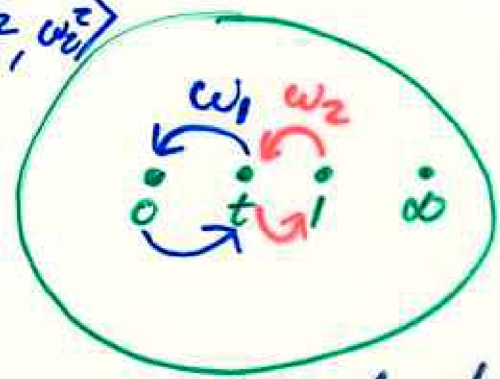
– universal family of cubic surfaces (Fricke-Klein/Cayley)

What is the monodromy action $\pi_1(B) \circlearrowleft M_{t_0}$?

$\pi_1(B) \cong F_2 \cong$ Pure mapping class group of $(\mathbb{P}^1, 0, t, 1, \infty)$

$\pi_1(B) = \langle l_1, l_2 \rangle \xrightarrow[l_i \mapsto \omega_i^2]{\sim} \langle \omega_1^2, \omega_2^2 \rangle = $ Pure mapping class group





Dehn twists

- Mapping class gp acts naturally on $M_t = \mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}), G)/G$ by "pushing forward loops"

$$\left[f(\rho)(\gamma) = \rho(f \circ \gamma) \qquad \begin{array}{l} \rho \in M_t \\ \gamma \in \pi_1 \\ f : \mathbb{P}^1 \to \mathbb{P}^1 \text{ diffeo} \end{array} \right]$$

- This is the monodromy action of π_1

- Explicitly on monod. matrices:

$$\omega_1(M_1, M_2, M_3) = (M_2, M_2 M_1 M_2^{-1}, M_3)$$

$$\omega_2(\text{———}) = (M_1, M_3, M_3 M_2 M_3^{-1})$$

Definition

Topological algebraic P_{VI} solution is an F_2 orbit of triples (M_1, M_2, M_3) of monodromy matrices which is finite

- clearly algebraic solutions have finite monodromy (& so finite F_2 orbits)

- In 2×2 case $M_i \in SL_2(\mathbb{C})$

- In 3×3 case M_i should be a pseudo-reflection (of form "1 + rank 1")

Obvious topological solutions:-

$\circledast$ [If M_1, M_2, M_3 generate a finite group then F_2 orbit must be finite] $\circledast$

Topological solution $\Rightarrow$ map $t : \Pi \to \mathbb{P}^1$
(topologically)

Idea: deg d rational maps $t : \Pi \to \mathbb{P}^1$
with r branch points

$$\Big\downarrow \cong \quad \text{(remove branchpoints)}$$

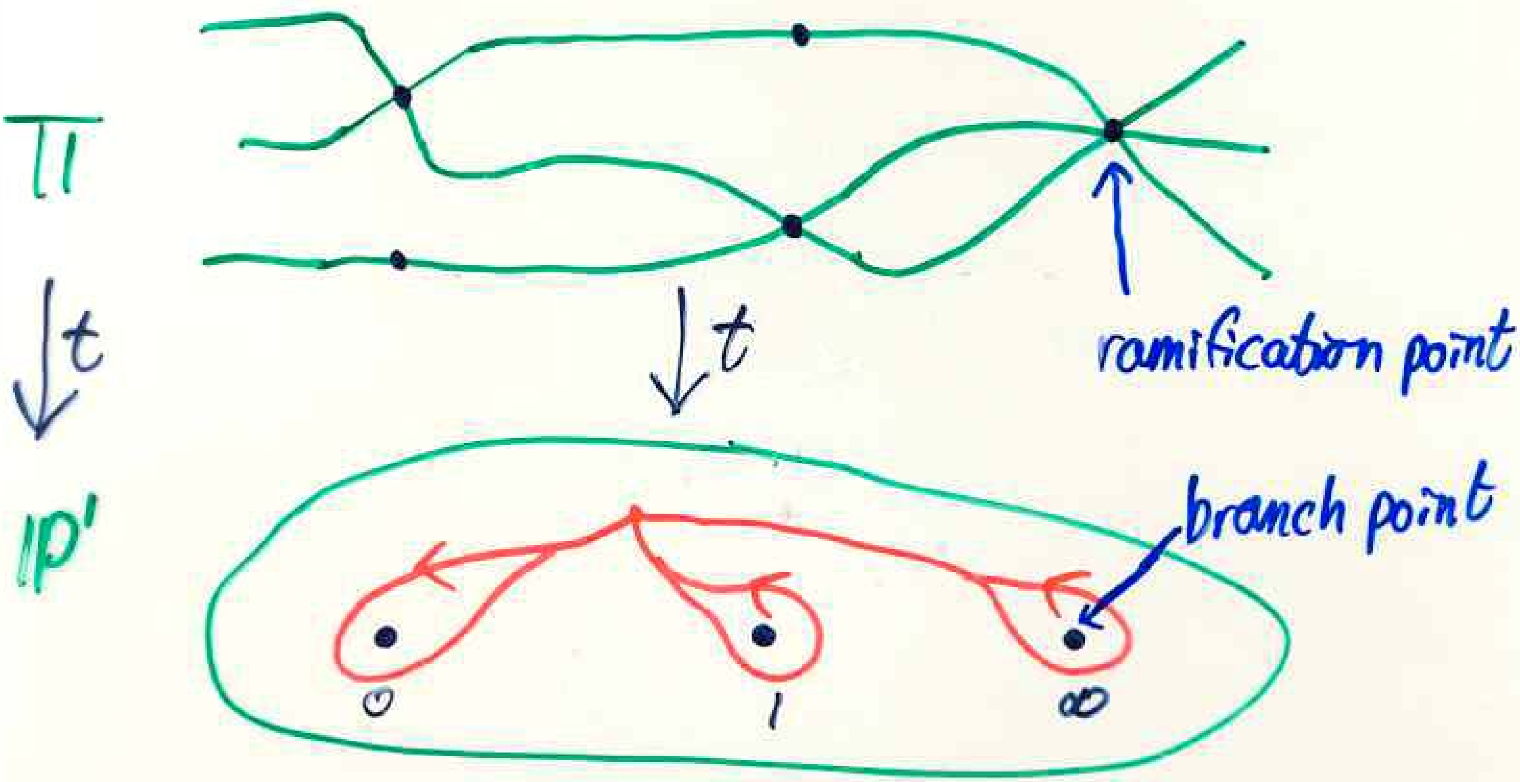
bundles over $\mathbb{P}^1 \setminus (r \text{ points})$
with fibres F = finite set with d-points

$$\Big\downarrow \cong \quad \text{(take monodromy)}$$

representations

$$\begin{array}{ccc} \pi_1(\mathbb{P}^1 \setminus r \text{ points}) & \longrightarrow & \mathrm{Aut}(F) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathcal{F}_{r-1} & \longrightarrow & \mathrm{Sym}\, d \end{array}$$

Here $r=3$ (Belyi map) so cover determined
by $\sigma_0, \sigma_1, \sigma_\infty \in \mathrm{Sym}\, d$
$(\sigma_\infty \sigma_1 \sigma_0 = 1)$



For us: fibre of t is $F = \left\{ F_2 \text{ orbit of } (M_1, M_2, M_3) \right\} / \text{conjugation}$

& F_2 action on F gives σ_0, σ_1

Riemann – Hurwitz $\Rightarrow$ genus of Π:

$$2g(\Pi) - 2 = d(2g(\mathbb{P}^1) - 2) + \sum_{\substack{\text{ramification}\\ \text{points}}} (i_r - 1)$$

i.e. $$g(\Pi) = 1 - d + \tfrac{1}{2}\sum (i_r - 1)$$

$= 1 - 3 + 2 = 0$ in example

[Ramification indices = cycle lengths of permutations $\sigma_0, \sigma_1, \sigma_\infty$

Can now go through list of triples of generators of

- finite subgroups $SL_2(\mathbb{C})$
- 3d complex reflection groups (generators should be reflections)

- Started by Hitchin

$SL_2(\mathbb{C})$, triples of form (g, g^{-1}, hgh^{-1})

generating binary dihedral, tetra-, octahedral gps

- All interesting dihedral solutions [Poncelet problem]
- 3/4 branch tet/oct solutions

- Effectively Dubrovin–Mazzocco did case of 3d real reflection group (orthogonal)

- 3/4 branch tet/oct solns (equiv. to Hitchin's)
- 3 icosahedral solns $d = 10, 10, 18$; $g = 0, 0, 1$

10 pages!

(equiv. to solutions w. finite $SL_2(\mathbb{C})$ monodromy)

$\theta = (0,0,0,k) \sim (k,k,k,k)/2$

Dihedral solutions

- interesting solutions have $\theta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
- equivalent to P_{VI} equation completely solved by Picard / R. Fuchs
- transcendental formula

Qn: explicit algebraic formula for Π, y, t?

- solved by Hitchin '96 (determinental formula) using Cayley's solution of Poncelet problem, Barth-Michel's parameterisation of modular curves...
- e.g D_5 – first explicit elliptic solution

[later M. Mazzocco looked at dihedral reflection groups – solutions equiv. to those above.

Elliptic dihedral solution

Hitchin 1996

12 branches, genus 1

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 1/2)$$

$$y = \frac{(3\,s-1)\,(s^2-4\,s-1)\,(s^2+u)\,(s\,(s+2)-u)}{(3\,s^3+7\,s^2+s+1)\,(s^2-u)\,(s\,(s-2)+u)}$$

$$t = \frac{\left(s^2+u\right)^2(s\,(s+2)-u)\,(s\,(s-2)-u)}{\left(s^2-u\right)^2(s\,(s+2)+u)\,(s\,(s-2)+u)}$$

where s, u satisfy:

$$u^2 = s\left(s^2+s-1\right)$$

(Triply generated) 3d complex reflection groups

Dihedral

Tetrahedral

Octahedral

Icosahedral

$G(m, p, 3)$

Klein $\qquad 2\,PSL_2(7)$

Hesse 1

Hesse 2

Valentiner $\qquad 6\,A_6$

[Shephard–Todd 1954]

Topological Klein solution

- upto equivalence Klein reflection group has 1 triple of generating reflections:

$$r_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & a \\ -1 & 1 & a \\ \bar{a} & \bar{a} & 0 \end{pmatrix},\ r_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},\ r_3 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

where $a = \frac{1}{2}(-1 + i\sqrt{7})$

- F_2 orbit (on conjugacy classes of triples) has size 7 — so degree $d = 7$

- permutation $\sigma_0, \sigma_1, \sigma_\infty$ each have cycle type $(2, 2, 3)$

  - so genus $(\Pi) = 1 - 7 + 3 \cdot 4/2 = 0$

- construction?

Similarly Valentiner group $\rightsquigarrow$ 3 genus 1 solutions degrees $= 15, 15, 24$

Existing construction methods

(1) "Algebraic geometry"

- Hitchin: find (solved) classical problem
- Dubrovin: Frobenius manifolds

(2) Pullbacks — Kitaev / Doran

- need ansatz & hope computer can find parameterized solution of N algebraic equations in $N+1$ unknowns (Kitaev has some examples)

(3) Exact asymptotics

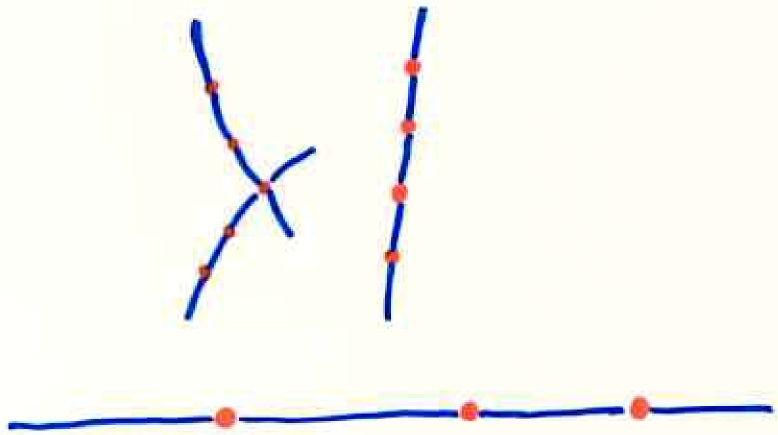
- Dubrovin-Mazzocco on line $\theta = (0,0,0,k)$ (proved asymptotic formula for such θ's) + classified all such solutions (5 + dihedral)

Key inputs

1. Jimbo

- leading asymptotics of P_{VI} solution y in terms of linear monodromy (M_1, M_2, M_3)

Degenerate to stable curve & solve Riemann-Hilbert problems there:



$\bar{B} = \bar{\mathcal{M}}_{0,4} = \mathbb{P}^1$

Theorem. (Jimbo 1982)

Suppose we have four matrices $M_j \in \mathrm{SL}_2(\mathbb{C})$, $j = 1, 2, 3, 4$ satisfying

a) $M_4 M_3 M_2 M_1 = 1$,

b) M_j has eigenvalues $\{\exp(\pm\pi i\theta_j)\}$ with $\theta_j \notin \mathbb{Z}$,

c) $\mathrm{Tr}(M_1 M_2) = 2\cos(\pi\sigma)$ for some nonzero $\sigma \in \mathbb{C}$ with $0 \le \mathrm{Re}(\sigma) < 1$,

d) None of the eight numbers

$$\theta_1 \pm \theta_2 \pm \sigma, \quad \theta_1 \pm \theta_2 \mp \sigma, \quad \theta_4 \pm \theta_3 \pm \sigma, \quad \theta_4 \pm \theta_3 \mp \sigma$$

is an even integer.

Then the leading term in the asymptotic expansion at zero of the corresponding Painlevé VI solution $y(t)$ on the branch corresponding to $[(M_1, M_2, M_3)]$ is

$$\left(\frac{(\theta_1+\theta_2+\sigma)(-\theta_1+\theta_2+\sigma)(\theta_4+\theta_3+\sigma)}{4\sigma^2(\theta_4+\theta_3-\sigma)\widehat{s}}\right) t^{1-\sigma}$$

where

$$\widehat{s} = c \times s, \qquad s = \frac{a+b}{d}$$

$$a = e^{\pi i\sigma}(i\sin(\pi\sigma)\cos(\pi\sigma_{23}) - \cos(\pi\theta_2)\cos(\pi\theta_4) - \cos(\pi\theta_1)\cos(\pi\theta_3))$$

$$b = i\sin(\pi\sigma)\cos(\pi\sigma_{13}) + \cos(\pi\theta_2)\cos(\pi\theta_3) + \cos(\pi\theta_4)\cos(\pi\theta_1)$$

$$d = 4\sin\left(\frac{\pi}{2}(\theta_1+\theta_2-\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_1-\theta_2+\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_4+\theta_3-\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_4-\theta_3+\sigma)\right)$$

$$c = \frac{(\Gamma(1-\sigma))^2\,\widehat{\Gamma}(\theta_1+\theta_2+\sigma)\,\widehat{\Gamma}(-\theta_1+\theta_2+\sigma)\,\widehat{\Gamma}(\theta_4+\theta_3+\sigma)\,\widehat{\Gamma}(-\theta_4+\theta_3+\sigma)}{(\Gamma(1+\sigma))^2\,\widehat{\Gamma}(\theta_1+\theta_2-\sigma)\,\widehat{\Gamma}(-\theta_1+\theta_2-\sigma)\,\widehat{\Gamma}(\theta_4+\theta_3-\sigma)\,\widehat{\Gamma}(-\theta_4+\theta_3-\sigma)}$$

where $\widehat{\Gamma}(x) := \Gamma(\frac{1}{2}x+1)$ (with Γ being the usual gamma function) and where $\sigma_{jk} \in \mathbb{C}$ $(j, k \in \{1, 2, 3\})$ is determined by $\mathrm{Tr}(M_j M_k) = 2\cos(\pi\sigma_{jk})$, $0 \le \mathrm{Re}(\sigma_{jk}) \le 1$, so $\sigma = \sigma_{12}$.

(2) Relate systems (A) & (B) on both DeRham & Betti sides

- monodromy changes in highly non-trivial way:

Klein reflection group $\rightsquigarrow$ Δ_{237}

Valentiner group $\rightsquigarrow$ A_5

- apparently procedure is complex analytic version of N. Katz's "middle convolution functor"

<u>Sketch</u> [Baber et al 1981, — Proc LMS 2005]

Fix distinct $a_1, \dots, a_n \in \mathbb{C}$

Let $A_0 = \mathrm{diag}(a_1, \dots, a_n)$

$$\begin{tikzcd}[column sep=small]
 & A \in \mathfrak{gl}_n(\mathbb{C}) \arrow[dl, "A_i = E_i A"'] \arrow[dr] & \\
d - \sum_1^n \frac{A_i}{z-a_i} dz \arrow[rr, leftrightarrow, "\text{Fourier-Laplace}"'] \arrow[d, "\text{Riemann-Hilbert}"] & & d - \left(\frac{A_0}{z^2} + \frac{A}{z}\right) dz \arrow[d, "\text{Stokes}"'] \\
\begin{array}{c} r_1, \dots, r_n \\ r_i = 1 + e_i \otimes \alpha_i \end{array} \arrow[rr, leftrightarrow, "\text{Killing–Coxeter}", "(\alpha_i(e_j)) = tu_+ - u_-"'] \arrow[dr, "r_n \cdots r_1"'] & & u_+, u_-, t \in U_+ \times U_- \times T \arrow[dl, "u_-^{-1} t u_+"] \\
 & G &
\end{tikzcd}$$

Scalar shift $A \mapsto A + \lambda I$

- tensor by $\lambda \frac{dz}{z}$ on RHS
- nontrivial convolution on LHS

$n = 3$: choose λ s.t. $A + \lambda$ rank $2 \Rightarrow$ reducible on LHS

- take 2×2 quotient connection $\rightsquigarrow$ SL_2 connection

$\Rightarrow$ $\mathbb{Z}_2$ equivariant maps: 3×3 triples $\mapsto$ 2×2 triples

3d reflection group	Subgroup $Sl_2(\mathbb{C})$
Tetrahedral	Octahedral
Octahedral	Tetrahedral
Icosahedral ($d=10,10,18$)	Icosahedral
Klein	Δ_{237}
Valentiner ($d=15,15,24$)	Icosahedral (!)

Topological solution

$\Downarrow$ Jimbo

Leading asymptotics at $t=0$ on each branch

$\Downarrow$ substitute back into P_{VI}

Any no. of terms of Puiseux expansion at 0 of y on each branch

$\Downarrow$ Finite no. coeffs to determine

Solution polynomial $F(y,t)$

$\Downarrow$ Maple / help from M. van Hoeij

Parameterised solution Π, y, t

[useful tricks listed in math.DG/0501464 $\rightsquigarrow$ Bolibruch's volume]

Klein solution

seven branches

$$(\theta_1,\theta_2,\theta_3,\theta_4)=(2/7,2/7,2/7,4/7)$$

$$y=-\frac{\left(5\,s^2-8\,s+5\right)\left(7\,s^2-7\,s+4\right)}{s\,(s-2)\,(s+1)\,(2\,s-1)\left(4\,s^2-7\,s+7\right)}$$

$$t=\frac{\left(7\,s^2-7\,s+4\right)^2}{s^3\left(4\,s^2-7\,s+7\right)^2}.$$

Corollary

For any s such that $t(s)\neq 0,1,\infty$ the family of Fuchsian systems

$$\frac{d}{dz}-\left(\frac{B_1}{z}+\frac{B_2}{z-t(s)}+\frac{B_3}{z-1}\right)$$

has monodromy isomorphic to the Klein complex reflection group, where

$$B_1=\begin{pmatrix}\frac12 & b_{12} & b_{13}\\ 0&0&0\\ 0&0&0\end{pmatrix},\qquad B_2=\begin{pmatrix}0&0&0\\ b_{21}&\frac12&b_{23}\\ 0&0&0\end{pmatrix},\qquad B_3=\begin{pmatrix}0&0&0\\ 0&0&0\\ b_{31}&b_{32}&\frac12\end{pmatrix}$$

$$b_{12}=\frac{14\,s^3-21\,s^2+24\,s-22}{21s\left(4\,s^2-7\,s+7\right)},\qquad b_{13}=\frac{22\,s^3-24\,s^2+21\,s-14}{21(7\,s^2-7\,s+4)},$$

$$b_{21}=\frac{14\,s^3-21\,s^2+24\,s+5}{21\,(s-1)\left(4\,s^2-s+4\right)},\qquad b_{23}=\frac{22\,s^3-42\,s^2+39\,s-5}{21(7\,s^2-7\,s+4)},$$

$$b_{31}=\frac{14-21\,s+24\,s^2+5\,s^3}{21\,(s-1)\left(4\,s^2-s+4\right)},\qquad b_{32}=\frac{22-42\,s+39\,s^2-5\,s^3}{21s\left(4\,s^2-7\,s+7\right)}.$$

Icosahedral Classification

Γ = binary icosahedral group $\subset SL_2(\mathbb{C})$

Prop. (P. Hall)

Up to conjugacy Γ has 26,688 triples of generators

Problem: Classify topological icosahedral solutions upto Okamoto's $W_a(F_4)$ action

Trick: bound above & below

Let $S = \{ (M_1, M_2, M_3) \mid M_i \in \Gamma, \langle M_i \rangle = \Gamma \} / \Gamma$

(so $\# S = 26688$)

Have map to θ-parameters

$$S \xrightarrow{p} \mathbb{Q}^4 \subset \mathbb{R}^4$$

$$(M_1, M_2, M_3) \longmapsto (\theta_1, \theta_2, \theta_3, \theta_4)$$

s.t M_j has eigenvalues $\exp(\pm \pi i \theta_j)$
& $\theta_j \in [0,1]$, $M_4 = (M_3 M_2 M_1)^{-1}$

Definition $T_1, T_2 \in S$ are parameter equivalent if $p(T_1)$ & $p(T_2)$ are in the same orbit of the standard $W_a(F_4)$ action (on $\mathbb{R}^4$)

Proposition
- Okamoto equivalence $\Rightarrow$ parameter equivalence
- S maps to exactly 52 parameter equivalence classes

Geometric Equivalence

Recall $\mathcal{F}_2 = \pi_1(B) \cong$ pure mapping class group of $\mathbb{P}^1$ w. 4 marked pts

- acts on S

Let MC = full mapping class group

$$1 \longrightarrow \mathcal{F}_2 \longrightarrow MC \longrightarrow Sym_4 \longrightarrow 1$$

- MC also acts on S

Also let $\Sigma = (\mathbb{Z}/2)^3 = \{(\pm 1, \pm 1, \pm 1)\}$

- acts on S in obvious way $(n_i \mapsto \pm n_i)$

$\Rightarrow$ group $\widetilde{MC} := MC \ltimes \Sigma$ acts on S

Def^n: orbits of $\widetilde{MC}$ = geometric equivalence classes

Prop.
- Geom. equiv. $\Rightarrow$ Okamoto equiv (delicate)
- $\widetilde{MC}$ has exactly 52 orbits on S

Corollary $\exists$ exactly 52 inequivalent icosahedral solutions to P_{VI} [–, Crelle 596 '06]

Icosahedral solutions with $\leqslant 4$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)	
1	1	0	1	$a\,b\,c$	31, 19, 11, 1	192	1	Schwarz $(y=t)$
2	1	0	1	$a\,b\,d$	37, 17, 13, 7	192	1	
3	1	0	1	$a\,c\,d$	33, 21, 9, 3	192	1	
4	1	0	1	$b\,c\,d$	28, 16, 8, 4	192	1	
5	1	0	2	$b^2\,c$	26, 14, 6, 6	96	1	
6	1	0	2	$b^2\,d$	38, 18, 18, 2	96	1	
7	1	0	2	$b\,c^2$	22, 10, 10, 2	96	1	
8	1	0	2	$b\,d^2$	34, 14, 10, 10	96	1	
9	1	0	3	c^3	18, 6, 6, 6	32	1	
10	1	0	3	d^3	42, 18, 18, 6	32	1	
11	2	0	2	$b^2\,c^2$	42, 18, 10, 10	96	2	$\sqrt{t}$
12	2	0	2	$b^2\,d^2$	50, 10, 6, 6	96	2	
13	2	0	2	$c^2\,d^2$	42, 18, 6, 6	96	2	
14	3	0	1	$b\,c^2\,d$	40, 16, 8, 8	288	S_3	Tet. family
15	3	0	1	$b\,c\,d^2$	40, 8, 4, 4	288	S_3	
16	4	0	2	$a\,c^3$	33, 9, 9, 9	128	A_4	Dih. family
17	4	0	2	$a\,d^3$	51, 3, 3, 3	128	A_4	
18	4	0	2	$c^3\,d$	30, 6, 6, 6	128	A_4	Oct. family
19	4	0	2	$c\,d^3$	42, 6, 6, 6	128	A_4	

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)	
20	5	0	1	$b^2 c d$	44, 12, 12, 4	480	S_5	
21	5	0	2	$c^2 d^2$	36, 12, 0, 0	240	S_5	Kitaev
22	6	0	1	$b c^2 d$	34, 10, 2, 2	576	S_6	
23	6	0	1	$b c d^2$	46, 14, 10, 2	576	S_6	
24	8	0	1	$a c^2 d$	39, 15, 3, 3	768	A_8	
25	8	0	1	$a c d^2$	45, 9, 9, 3	768	A_8	
26	9	1	2	$b c^3$	28, 4, 4, 4	288	A_9	Kitaev
27	9	1	2	$b d^3$	52, 8, 8, 4	288	A_9	
28	10	0	2	$a^2 c d$	48, 12, 6, 6	480	$2^7\,3\,5$	
29	10	0	2	$b^3 c$	46, 14, 14, 6	320	A_{10}	
30	10	0	2	$b^3 d$	42, 2, 2, 2	320	A_{10}	
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}	DM
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}	DM
33	12	0	0	$a b c d$	43, 11, 7, 1	2304	A_{12}	
34	12	1	1	$a b c^2$	37, 13, 5, 5	1152	A_{12}	
35	12	1	1	$a b d^2$	49, 5, 5, 1	1152	A_{12}	
36	12	1	1	$b^2 c d$	38, 6, 6, 2	1152	$2^9\,3^2\,5$	
37	15	1	2	$b^3 c$	36, 4, 4, 4	480	A_{15}	Valentiner
38	15	1	2	$b^3 d$	48, 8, 8, 8	480	A_{15}	Valentiner
39	15	1	2	$b^2 c^2$	32, 8, 0, 0	720	S_{15}	
40	15	1	2	$b^2 d^2$	44, 4, 0, 0	720	S_{15}	
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$	DM
42	20	1	1	$a b^2 c$	41, 9, 9, 1	1920	A_{20}	
43	20	1	1	$a b^2 d$	47, 7, 3, 3	1920	A_{20}	
44	20	1	3	$a^2 c^2$	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$	
45	20	1	3	$a^2 d^2$	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$	
46	24	1	2	$a b^3$	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$	Valentiner
47	30	2	2	$a^2 b c$	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	
48	30	2	2	$a^2 b d$	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	
49	36	3	3	$a^2 b^2$	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$	
50	40	3	3	$a^3 c$	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$	
51	40	3	3	$a^3 d$	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$	
52	72	7	3	$a^3 b$	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$	

Solution 20, genus zero, 5 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/3, 1/5, 2/3)$:

$$y = \frac{2\,(s^2+s+7)\,(5\,s-2)}{s\,(s+5)\,(4\,s^2-5\,s+10)}, \qquad t = \frac{27\,(5\,s-2)^2}{(s+5)\,(4\,s^2-5\,s+10)^2}$$

Solution 24, genus zero, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 2/5, 1/5, 4/5)$:

$$y = \frac{s\,(s+4)\,(3\,s^4-2\,s^3-2\,s^2+8\,s+8)}{8\,(s-1)\,(s^2+4)\,(s+1)^2}, \qquad t = \frac{s^5\,(s+4)^3}{4\,(s-1)\,(s^2+4)^2\,(s+1)^3}$$

Solution 25, genus zero, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 2/5, 1/2, 4/5)$:

$$y = \frac{s^2\,(5\,s^3+2\,s^2-4\,s-8)\,(s+4)^2}{4\,(s+1)^2\,(s^2+4)\,(s-1)\,(s^2+3\,s+6)}, \qquad t = \frac{s^5\,(s+4)^3}{4\,(s-1)\,(s^2+4)^2\,(s+1)^3}$$

Solution 28, genus zero, 10 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/5, 3/5)$:

$$y = \frac{(s^5+5\,s^4-20\,s^3+75\,s+75)\,(s^2-5)\,(s^2+5)}{(s+1)^2\,(s^2-4\,s+5)\,(s+5)\,(s^4+6\,s^2-75)}, \quad t = \frac{2\,(s^2+5)^3\,(s^2-5)^2}{(s+5)^3\,(s^2-4\,s+5)^2\,(s+1)^3}$$

"Generic" solution, genus zero, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/2, 1/3, 4/5)$:

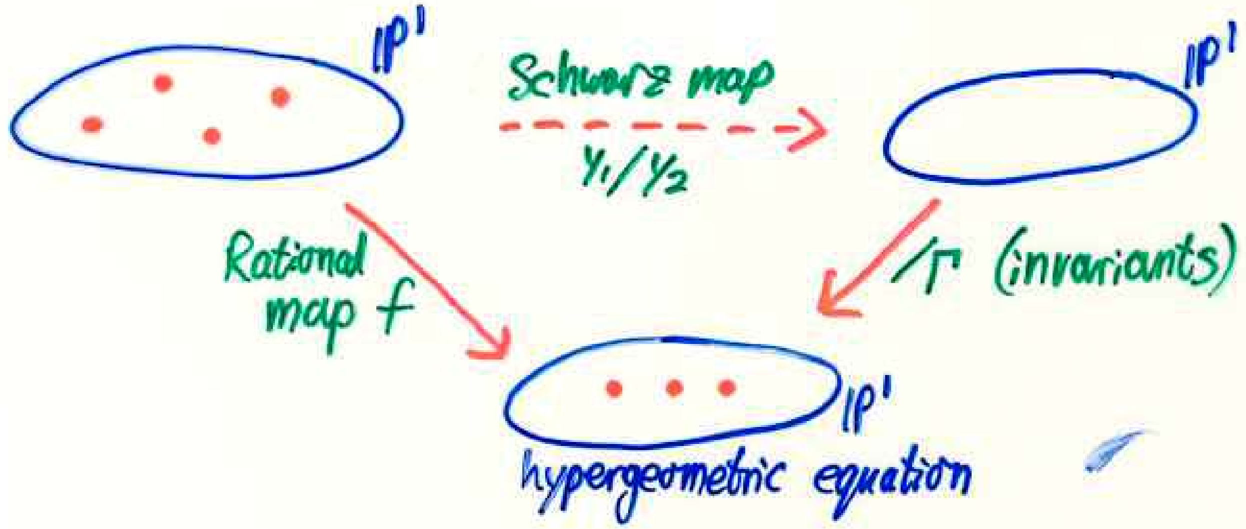
$$y = -\frac{9s(s^2+1)(3s-4)(15s^4-5s^3+3s^2-3s+2)}{(2s-1)^2(9s^2+4)(9s^2+3s+10)}$$

$$t = \frac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2}$$

$$\begin{gathered}
F(y,t) = \\
(15524784\,t^2 - 5373216\,t + 1350000)\,y^{12} - (128381760\,t^2 - 13366080\,t)\,y^{11} + \\
(5425704\,t^3 + 496677744\,t^2 - 30539160\,t)\,y^{10} - \\
(14929920\,t^4 + 41364000\,t^3 + 866759680\,t^2 - 2928160\,t)\,y^9 + \\
(107546535\,t^4 - 508275750\,t^3 + 747613335\,t^2 - 1837080\,t)\,y^8 - \\
(24385536\,t^5 - 285548724\,t^4 - 2437066824\,t^3 + 74927724\,t^2 + 944784\,t)\,y^7 + \\
(58212000\,t^5 - 2865570750\,t^4 - 4456260900\,t^3 + 17631810\,t^2)\,y^6 - \\
(49787136\,t^6 - 904003584\,t^5 - 7215732804\,t^4 - 2130570936\,t^3 - 12872196\,t^2)\,y^5 - \\
(413500320\,t^6 + 3724484160\,t^5 + 4839581265\,t^4 + 162430110\,t^3 + 3750705\,t^2)\,y^4 + \\
(3001304640\,t^6 + 74794560\,t^5 + 2710584000\,t^4 - 380946240\,t^3)\,y^3 - \\
(940800000\,t^7 + 977540640\,t^6 - 726801696\,t^5 + 939255264\,t^4 - 72013536\,t^3)\,y^2 + \\
(1176000000\,t^7 - 1481095680\,t^6 + 765158400\,t^5)\,y - \\
(1920800000\,t^8 - 7212800000\,t^7 + 10522980864\,t^6 - 6913299456\,t^5 + 1728324864\,t^4)
\end{gathered}$$

Pullbacks (Klein, R-Fuchs, ..., Kitaev, C. Doran, ...)

Klein showed all 2nd order Fuchsian equations with finite monodromy are (essentially) pullbacks of hypergeometric equations:



so isomonodromic family of ODEs ~ family of rational maps

Key observation: algebraicity of deformation comes from that of rational maps (Hurwitz spaces)
(Doran, Kitaev)
not from finiteness of monodromy representation

C. Doran JDG 2001

regular singular point at λ, and precisely four non-apparent regular singular points at $\{0, 1, \infty, t\}$. The local monodromies about these points do not vary with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. By Lemma 2.9, we thus know that λ as a function of t determines a solution to a Painlevé VI equation as described. q.e.d.

A direct application of this criterion to the natural hypergeometric local systems associated to triangles yields the following three corollaries:

Ramification indices over $0, 1, \infty$

Corollary 4.6. *The following is the complete list of topological types corresponding to algebraic Painlevé VI solutions coming from pullback from arithmetic Fuchsian triangle groups, together with the description of the corresponding triangle:*

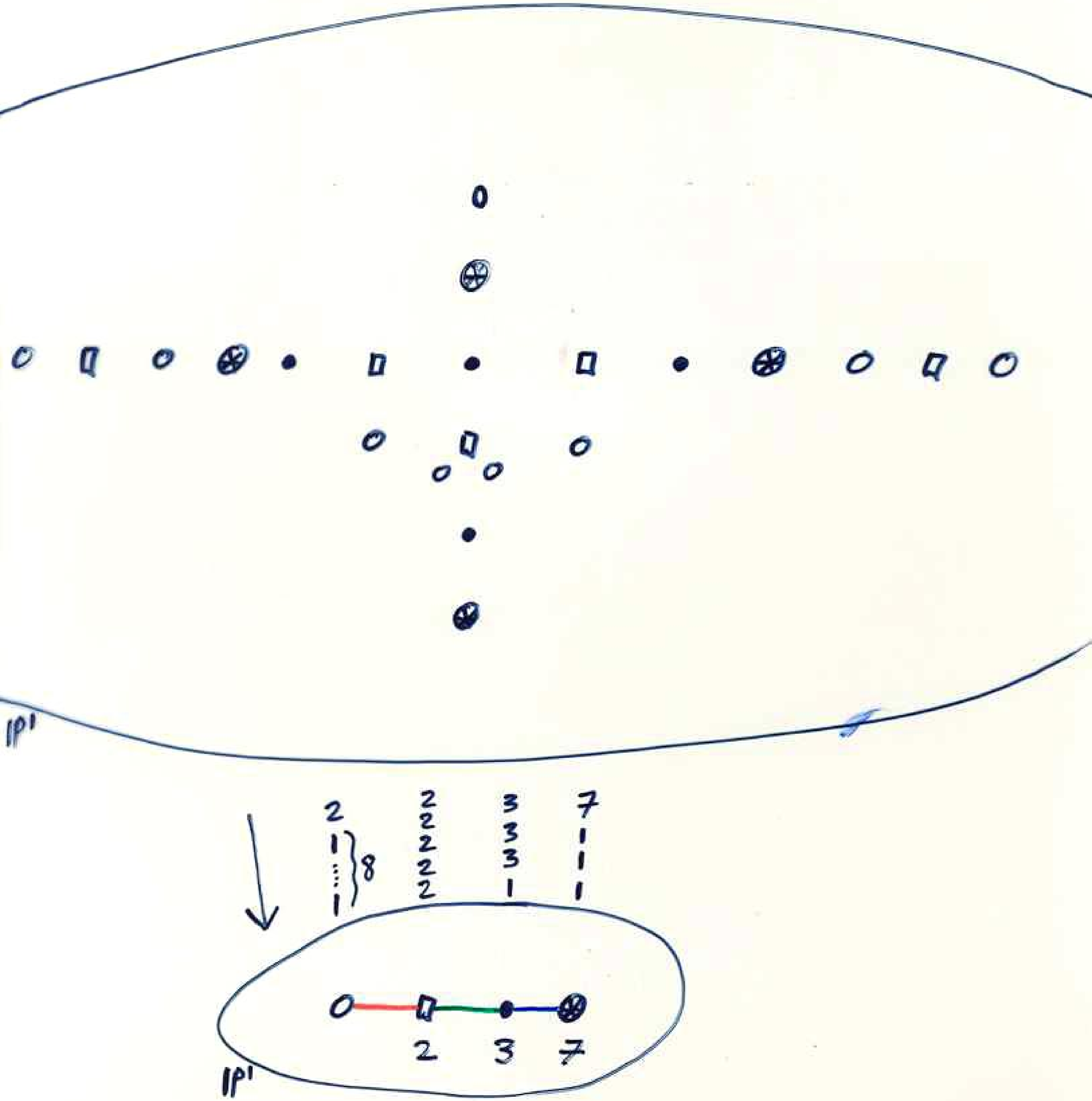
Degree of rational map f

	Triangle group	
$(2; [2], [1, 1], [1, 1]; 2)$	$(2, \square, \square)$	
$(3; [2, 1], [3], [1, 1, 1]; 2)$	$(2, 3, \square)$	
$(4; [2, 2], [3, 1], [2, 1, 1]; 2)$	$(2, 3, \square)$	≤ 4 branches
$(4; [2, 2], [4], [1, 1, 1, 1]; 2)$	$(2, 4, \square)$	
$(6; [2, 2, 2], [3, 3], [2, 2, 1, 1]; 2)$	$(2, 3, \square)$	
$(6; [2, 2, 2], [3, 3], [3, 1, 1, 1]; 2)$	$(2, 3, \square)$	
$(10; [2, \ldots, 2], [3, 3, 3, 1], [7, 1, 1, 1]; 2)$	$(2, 3, 7)$	$g=1$, $d=18$ new
$(12; [2, \ldots, 2], [3, 3, 3, 3], [7, 2, 1, 1, 1]; 2)$	$(2, 3, 7)$	Klein
$(12; [2, \ldots, 2], [3, 3, 3, 3], [8, 1, 1, 1, 1]; 2)$	$(2, 3, 8)$	$\sqrt{t}$ or Octahedral
$(18; [2, \ldots, 2], [3, \ldots, 3], [7, 7, 1, 1, 1, 1]; 2)$	$(2, 3, 7)$	$\sqrt{t}$

Here $\square$ represents any of the possible entries as listed in Theorem 4.4.

Note that in the case of the arithmetic triangle group $\mathrm{PSL}(2, \mathbb{Z})$, with triangle $(2, 3, \infty)$, as expected we recover from this list the topological types of the Kodaira functional inveriants of our five families. In this corollary, the restriction to arithmetic Fuchsian triangle groups is for convenience only — we just wanted a finite set of triangle groups in $\mathrm{PSL}(2, \mathbb{R})$ to which to apply our criterion, and in this case they yielded a finite list of topological types. By contrast, for some triangles one can explicitly construct infinite lists of allowable topological types (unlike the previous result, the proofs of these corollaries do not produce an exhaustive list of types, merely an infinite one):

Corollary 4.7. *There are infinitely many topological types corresponding to algebraic Painlevé VI solutions arising by pullback from each triangle uniformized by $\mathbb{C}$, except for $(3, 3, 3)$ which has none.*



ℙ¹
2
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3
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3
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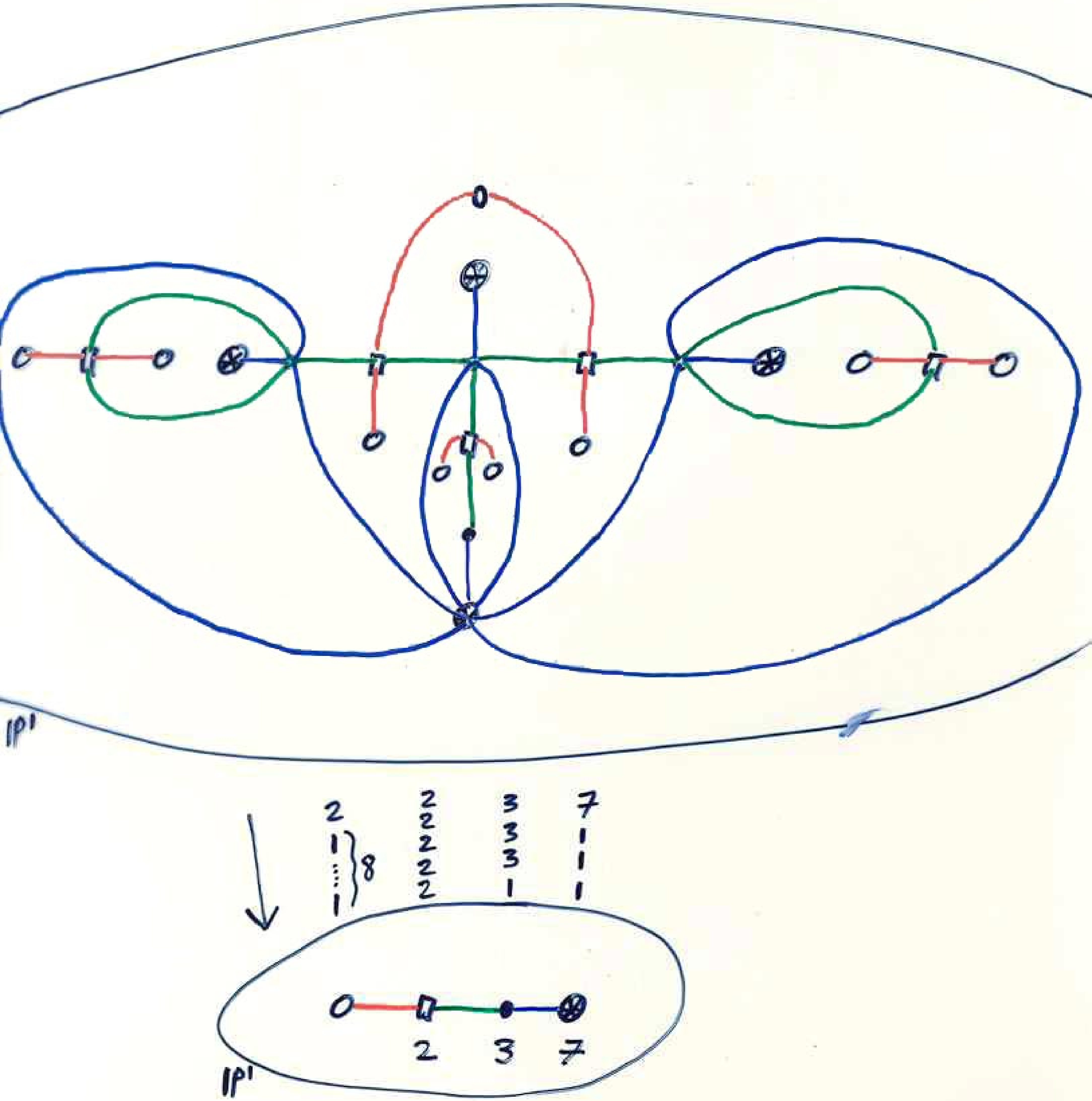
ℙ¹



IP'
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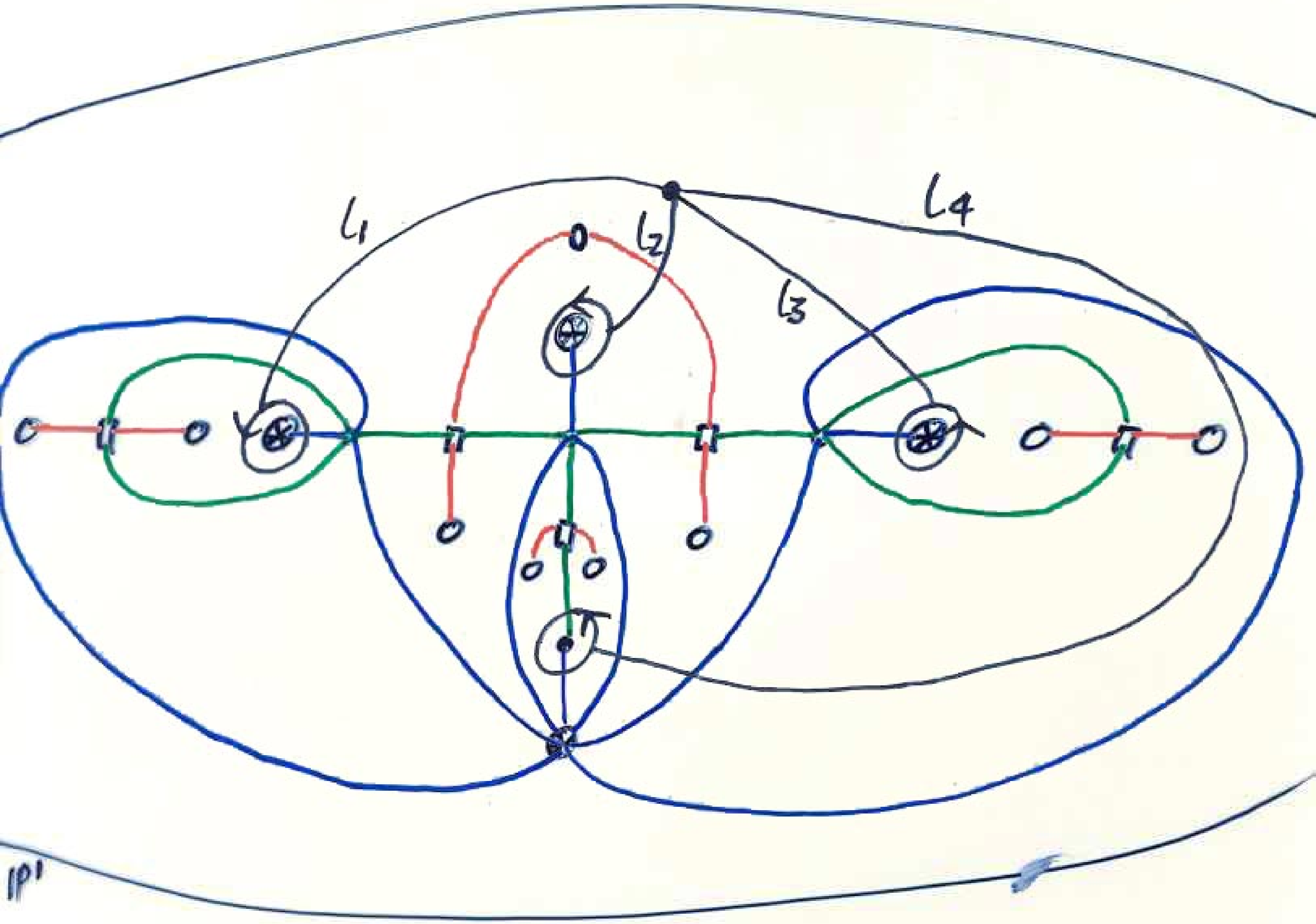
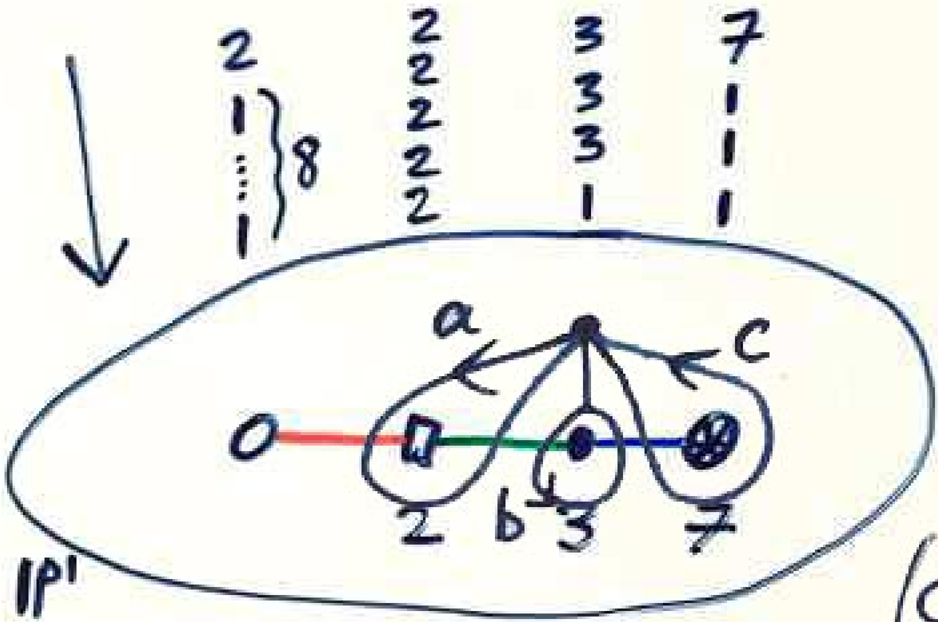




$$l_1 = c a c a^{-1} c^{-1}$$
$$l_2 = c$$
$$l_3 = c^{-1} a^{-1} c a c$$
$$l_4 = c^{-3} b c^{3}$$

$(cba = 1)$

Simple observation

Can write down topological P_{VI} solution
from topology of f, by hand
(don't need f explicitly)

- go through Doran's list & find top. solutions
- compute explicitly by previous asymptotic method.

2, 3, 7 solution

genus one, 18 branches

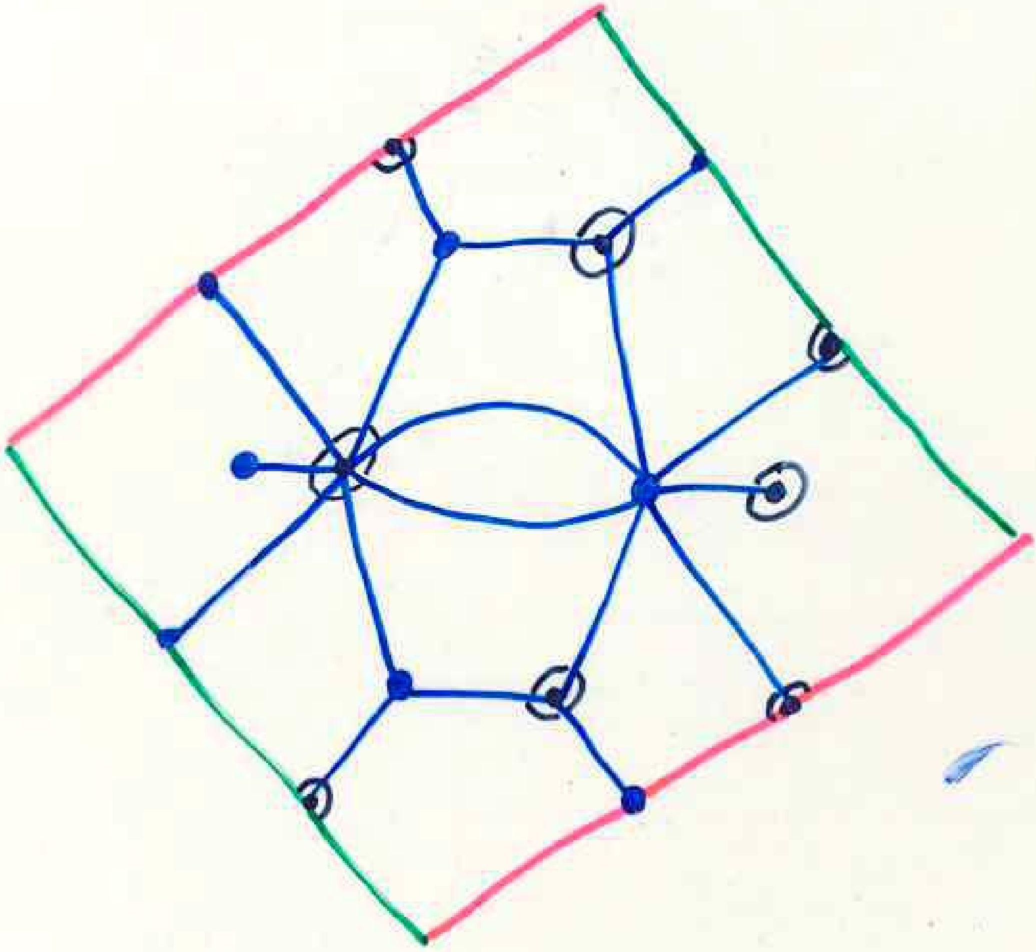
$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/7, 2/7, 2/7, 1/3)$

$$y = \frac{1}{2} - \frac{(3\,s^8 - 2\,s^7 - 4\,s^6 - 204\,s^5 - 536\,s^4 - 1738\,s^3 - 5064\,s^2 - 4808\,s - 3199)\,u}{4\,(s^6 + 196\,s^3 + 189\,s^2 + 756\,s + 154)\,(s^2 + s + 7)\,(s + 1)}$$

$$t = \frac{1}{2} - \frac{(s^9 - 84\,s^6 - 378\,s^5 - 1512\,s^4 - 5208\,s^3 - 7236\,s^2 - 8127\,s - 784)\,u}{432\,s\,(s+1)^2\,(s^2 + s + 7)^2}$$

where

$$u^2 = s\,(s^2 + s + 7).$$



– thanks to M. van Hoeij

Icosahedral solution 41

genus one, 18 branches

$(\theta_1,\theta_2,\theta_3,\theta_4) = (1/3,1/3,1/3,1/3)$

$$y = \frac{1}{2} - \frac{8\,s^7 - 28\,s^6 + 75\,s^5 + 31\,s^4 - 269\,s^3 + 318\,s^2 - 166\,s + 56}{18\,u\,(s-1)\,(3\,s^3 - 4\,s^2 + 4\,s + 2)}$$

$$t = \frac{1}{2} + \frac{(s+1)\,\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right)}{54\,u^3\,s\,(s-1)}$$

where $\qquad u^2 = s\,(8\,s^2 - 11\,s + 8).$

(Equivalent to Dubrovin–Mazzocco's 10 page elliptic solution.)

The corresponding family of connections on $\mathbb{P}^1$ with icosahedral monodromy is:

$$d - \left(\frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}\right) dz, \quad \text{where}$$

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

$$b_{12} = \lambda_1 - \mu_3 y + (\mu_1 - xy)(y-1), \qquad b_{32} = (\mu_2 - \lambda_2 - b_{12})/t,$$

$$b_{13} = \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), \qquad b_{23} = (\mu_2 - \lambda_3)t - b_{13},$$

$$b_{21} = \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, \qquad b_{31} = (\mu_2 - \lambda_1 - b_{21})/t$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}, \qquad \mu_1, \mu_2, \mu_3 = \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$$

$$x = \frac{24\,(s-1)\,(3\,s^3 - 4\,s^2 + 4\,s + 2)\,P(s)\,u}{5\,(6\,s^2 - 2\,s + 1)\,(4\,s^4 + 4\,s^3 + 54\,s^2 - 86\,s + 49)\,(2\,s-1)^2\,(2\,s^2 + s + 2)^2\,(s-2)^4}$$

$$P = 114\,s^9 - 416\,s^8 + 1184\,s^7 + 814\,s^6 - 6016\,s^5 + 9136\,s^4 - 6634\,s^3 + 2716\,s^2 - 364\,s + 91.$$

24 branch Valentiner solution
(Icosahedral Solution 46)

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/3, 1/3, 1/2)$$

$$y = \frac{1}{2} - \frac{\mathrm{P}}{2\,(3\,s^2 - 2\,s + 2)\,\mathrm{R}\,u}, \quad t = \frac{1}{2} + \frac{(s^2 + 4\,s - 2)\,\mathrm{Q}}{2\,(s+2)\,(3\,s^2 - 2\,s + 2)^2\,u^3}$$

where

$$\mathrm{P} = 16\,s^{11} + 72\,s^{10} + 50\,s^9 - 242\,s^8 - 3143\,s^7 + 6562\,s^6 - 8312\,s^5 + 9760\,s^4 - 9836\,s^3 + 6216\,s^2 - 2288\,s + 416,$$
$$\mathrm{Q} = 8\,s^{10} + 16\,s^9 + 24\,s^8 - 84\,s^7 + 429\,s^6 - 312\,s^5 + 258\,s^4 - 288\,s^3 + 288\,s^2 - 128\,s + 32,$$
$$\mathrm{R} = 26\,s^6 + 18\,s^5 - 75\,s^4 + 50\,s^3 + 270\,s^2 - 312\,s + 104,$$

and where (u, s) lies on the elliptic curve

$$u^2 = (8\,s^2 - 7\,s + 2)\,(s+2)\,.$$

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
20	5	0	1	b^2cd	44, 12, 12, 4	480	S_5
21	5	0	2	c^2d^2	36, 12, 0, 0	240	S_5
22	6	0	1	bc^2d	34, 10, 2, 2	576	S_6
23	6	0	1	bcd^2	46, 14, 10, 2	576	S_6
24	8	0	1	ac^2d	39, 15, 3, 3	768	A_8
25	8	0	1	acd^2	45, 9, 9, 3	768	A_8
26	9	1	2	bc^3	28, 4, 4, 4	288	A_9
27	9	1	2	bd^3	52, 8, 8, 4	288	A_9
28	10	0	2	a^2cd	48, 12, 6, 6	480	$2^7\,3\,5$
29	10	0	2	b^3c	46, 14, 14, 6	320	A_{10}
30	10	0	2	b^3d	42, 2, 2, 2	320	A_{10}
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}
33	12	0	0	$abcd$	43, 11, 7, 1	2304	A_{12}
34	12	1	1	abc^2	37, 13, 5, 5	1152	A_{12}
35	12	1	1	abd^2	49, 5, 5, 1	1152	A_{12}
36	12	1	1	b^2cd	38, 6, 6, 2	1152	$2^9\,3^2\,5$
37	15	1	2	b^3c	36, 4, 4, 4	480	A_{15}
38	15	1	2	b^3d	48, 8, 8, 8	480	A_{15}
39	15	1	2	b^2c^2	32, 8, 0, 0	720	S_{15}
40	15	1	2	b^2d^2	44, 4, 0, 0	720	S_{15}
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$
42	20	1	1	ab^2c	41, 9, 9, 1	1920	A_{20}
43	20	1	1	ab^2d	47, 7, 3, 3	1920	A_{20}
44	20	1	3	a^2c^2	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
45	20	1	3	a^2d^2	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
46	24	1	2	ab^3	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$
47	30	2	2	a^2bc	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
48	30	2	2	a^2bd	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
49	36	3	3	a^2b^2	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$
50	40	3	3	a^3c	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$
51	40	3	3	a^3d	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$
52	72	7	3	a^3b	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$

Quadratic / Landen / Folding transformations

Kitaev — Manin — Tsuda-Okamoto-Sakai

Kitaev's perspective:

If A a fuchsian system with poles at $0, t, 1, \infty$

& with (proj.) monodromy of order 2 at $0, \infty$

- pullback A along $z \longmapsto z^2$
- get system B with 4 non-apparent sing.s at $\pm 1, \pm\sqrt{t}$
- remove apparent sings & renormalize

* IMDs of A $\Leftrightarrow$ IMDs of resulting system *

$\rightsquigarrow$ get transform relating certain P_{VI} solutions (codim 2 in param. space)

- Much simpler explicit formulae for transform later (conjugate by Okamoto transformations) (Ramani, Grammaticos, Tamizhmani 2000)

Theorem (Ramani–Grammaticos–Tamizhmani 2000)
Given a solution (y_0, t_0) of P_{VI} with parameters of the form

$$\theta = (0, \theta_2, \theta_3, 1)$$

then, by taking two square roots, one obtains a new solution (y, t) with parameters

$$\theta = (\theta_3, \theta_2, \theta_2, 2 - \theta_3)/2$$

where

$$y = \frac{(\tau - 1)(\eta + 1)}{(\tau + 1)(\eta - 1)}, \qquad t = \left(\frac{\tau - 1}{\tau + 1}\right)^2$$

with

$$\eta^2 = y_0, \qquad \tau^2 = t_0.$$

Theorem$'$ (Tsuda–Okamoto–Sakai 2005)

Given a solution (y_0, t_0) of P_{VI} with parameters of the form

$$\theta = (\theta_1, \theta_2, \theta_2, 1 - \theta_1)$$

then, by taking one square root, one obtains a new solution (y, t) with parameters

$$\theta = (0, 2\theta_2, 0, 1 - 2\theta_1)$$

where

$$y = \frac{1}{2} + \frac{1}{4}\left(\frac{\tau}{y_0} + \frac{y_0}{\tau}\right)$$

$$t = \frac{1}{2} + \frac{1}{4}\left(\tau + \frac{1}{\tau}\right)$$

with

$$\tau^2 = t_0.$$

Corollary ("Unfolding transformation")
If functions y_0, t_0 of the form

$$y_0 = \frac{1}{2} + a_y(s)u, \qquad t_0 = \frac{1}{2} + a_t(s)u$$

are a $\mathrm{P_{VI}}$ solution with parameters

$$\theta = (0, \theta_2, 0, \theta_4)$$

on a Painlevé curve of the form

$$\Pi := \{u^2 = u_2(s)\}$$

for a polynomial $u_2(s)$, then the functions

$$y = \frac{1}{2} + \frac{w+v}{2(A_y - A_t)}, \qquad t = \frac{1}{2} - \frac{A_t}{2w}$$

are a $\mathrm{P_{VI}}$ solution for parameters

$$\theta = (1 - \theta_4, \theta_2, 1 - \theta_4, 2 - \theta_2)/2$$

on the curve obtained by adjoining to $\mathbb{C}(s)$ the functions v, w where

$$v^2 = A_y^2 - u_2, \qquad w^2 = A_t^2 - u_2$$

and $A_i = 2a_i u_2$ for $i = y, t$.

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
20	5	0	1	$b^2\,c\,d$	44, 12, 12, 4	480	S_5
21	5	0	2	$c^2\,d^2$	36, 12, 0, 0	240	S_5
22	6	0	1	$b\,c^2\,d$	34, 10, 2, 2	576	S_6
23	6	0	1	$b\,c\,d^2$	46, 14, 10, 2	576	S_6
24	8	0	1	$a\,c^2\,d$	39, 15, 3, 3	768	A_8
25	8	0	1	$a\,c\,d^2$	45, 9, 9, 3	768	A_8
26	9	1	2	$b\,c^3$	28, 4, 4, 4	288	A_9
27	9	1	2	$b\,d^3$	52, 8, 8, 4	288	A_9
28	10	0	2	$a^2\,c\,d$	48, 12, 6, 6	480	$2^7\,3\,5$
29	10	0	2	$b^3\,c$	46, 14, 14, 6	320	A_{10}
30	10	0	2	$b^3\,d$	42, 2, 2, 2	320	A_{10}
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}
33	12	0	0	$a\,b\,c\,d$	43, 11, 7, 1	2304	A_{12}
34	12	1	1	$a\,b\,c^2$	37, 13, 5, 5	1152	A_{12}
35	12	1	1	$a\,b\,d^2$	49, 5, 5, 1	1152	A_{12}
36	12	1	1	$b^2\,c\,d$	38, 6, 6, 2	1152	$2^9\,3^2\,5$
37	15	1	2	$b^3\,c$	36, 4, 4, 4	480	A_{15}
38	15	1	2	$b^3\,d$	48, 8, 8, 8	480	A_{15}
39	15	1	2	$b^2\,c^2$	32, 8, 0, 0	720	S_{15}
40	15	1	2	$b^2\,d^2$	44, 4, 0, 0	720	S_{15}
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$
42	20	1	1	$a\,b^2\,c$	41, 9, 9, 1	1920	A_{20}
43	20	1	1	$a\,b^2\,d$	47, 7, 3, 3	1920	A_{20}
44	20	1	3	$a^2\,c^2$	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
45	20	1	3	$a^2\,d^2$	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
46	24	1	2	$a\,b^3$	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$
47	30	2	2	$a^2\,b\,c$	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
48	30	2	2	$a^2\,b\,d$	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
49	36	3	3	$a^2\,b^2$	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$
50	40	3	3	$a^3\,c$	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$
51	40	3	3	$a^3\,d$	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$
52	72	7	3	$a^3\,b$	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$

Solution 52

72 branches, genus 7

$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/12, 1/12, 1/12, 11/12)$

$$y = \frac{1}{2} + \frac{9\,(j-1)\,(j^3+27j^2-57j+79)\,wv + 2\,(2j^2-2j+5)\,(j^2-7j+1)\,(2j^4+2j^3-3j^2-58j+107)\,(j^2-4j+13)^2}{6\,(j^2-1)\,(2j^2+j+17)\,(j^3-3j^2+3j-11)\,(2j-7)^2\,v}$$

$$t = \frac{1}{2} + \frac{(s+1)\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right)}{54\,s\,(s-1)\,u^3}$$

on the curve in $\mathbb{P}^3$ with affine equations:

$$v^2 = -(j+1)\left(6+j^2-2j\right)\left(4j^2-13j+19\right),$$
$$w^2 = (j-1)\,(2j-7)\,(j+1)\left(2j^2+j+17\right)\left(4j^2-13j+19\right)$$

where

$$s = \frac{j^2-1}{2j-7}, \qquad u = \frac{w}{(2j-7)^2}.$$

In fact this genus 7 curve is birational to the plane octic cut out by:

$$9\,(p^6q^2+p^2q^6)+18\,p^4q^4+4\,(p^6+q^6)+26\,(p^4q^2+p^2q^4)+8\,(p^4+q^4)+57\,p^2q^2+20\,(p^2+q^2)+16$$

Problems

- Prove there are no more algebraic solutions

- Is there another embedding of P_{VI} in the Schlesinger system s.t. the $g=1$ 237 solutions controls IMDs of fuchsian systems with finite monodromy?

- Why are the Painlevé curves Π defined $/\mathbb{Q}$?

- Extend Hitchin's twistor viewpoint to the icosahedral solutions
  - ~ rational curves in Umemura-Mukai's 3-fold